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SB-3453

M. A. / M. Sc. (Part I) Examination

March / April – 2011

Mathematics : Paper-405

(Ordinary Differential Equations)

(Old Course)

Time : Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **M. A. / M. Sc. (Part I)**

Name of the Subject :
☛ **Mathematics : Paper-405**

☛ Subject Code No. : **3 4 5 3** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Answer all question.

(3) Figures to the right indicate marks of the question.

(4) Follow usual notation.

1 (a) If g(t,u) is a continuous function of t and u in a closed, bounded region $R(a,b)$ and satisfies the Lipschitz's condition in R, then prove that, there exists a unique solution u(t) to the initial value problem u'=g(t,u), $u(t_0)=u_0$ defined on the interval J. **5**

(b) If $\Phi(t)$ is a fundamental matrix of $x' = Ax$ where $x \in R^n$, A is an $n \times n$ constant matrix and $\Phi(D) = I$, then $\Phi(t)\Phi(\alpha) = \Phi(t-\alpha)$ for every α. **5**

(c) Show that $g(t,u) = t^2u^2 + u^4$, $R: |t| \le 1$, $|u-2| \le 3$ satisfies the Lipschitz condition in the region R. And find the Lipschitz constant k. **4**

OR

1 (a) If C is a non negative constant and u, v are non negative continuous functions on some interval $t_0 \le t \le t_0 + a$ satisfying $u(t) \le c \exp\left[\int_{t_o}^{t} u(s)v(s)ds\right]$, then prove that $u(t) \le c \exp\left[\int_{t_0}^{t} v(s)ds\right]$ for $t \in [t_0, t_0 + a]$. **5**

(b) Prove that a fundamental matrix Φ of system $x' = Ax$ is given by $\Phi(t) = e^{+A}$, $-\infty < t < \infty$, A is constant matrix. Also show that any solution $x(t)$ of this system satisfying $x(t_0) = x_0$ is given by $x(t) = e^{+A}x_0$. **5**

(c) Apply Picard's method to solve the initial value problem $u' = 4t + 2tu$, $u(0) = 1$. 4

2 (a) If $\varphi(t)$ is a fundamental matrix of $x' = A(t)x$ then prove that $\det \Phi(t) = \det \Phi(t_0) \exp\left[\int_{t_0}^{t} T_0 A(s)ds\right]$ for $t \in (r_1, r_2)$. **5**

(b) If Φ is a fundamental matrix of $x' = A(t)x$, then ψ is a fundamental matrix of its adjoint system if and only if $\psi^T \Phi = c$ for some constant non-singular matrix C. **5**

(c) Determine the stable characteristic polynomial for the equation $y''' + 2y'' + 3y' + 4y = 0$. 4

OR

2 (a) For the equation $x' = A(t)x + B(t)$; $x(t_0) = x_0$, $t_0 \in (\gamma_1, \gamma_2)$ **5**

show that the solution $x(t)$ is given by

$x(t) = \Phi(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds$ where $\Phi(t)$ is a fundamental matrix satisfying $\Phi(t) = I$.

(b) Let $P(t)$ and $q(t)$ be the continuous function of t in (γ_1, γ_2). If $y_1(t)$ and $y_2(t)$ are the solutions of the differential equation $y'' + P(t)y' + q(t)y = 0$ satisfying the condition $W_{12}(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$ at every point in the interval (γ_1, γ_2) then show that any other solution of this equation is a linear combination of y_1 and y_2. **5**

(c) Determine whether all the solution $y(t)$ of $y^{(4)} + 4y^{(3)} + 2y'' + 6y' + 2y = 0$ approach zero or not as $t \to \infty$. 4

3 (a) Prove that $x' = A(t)x$ is strongly stable if and only if there exists a positive constant M such that $\|\Phi(t)\| \leq M$, $\|\Phi^{-1}(t)\| \leq M$ for $t \geq t_0$, where $\Phi(t)$ is a fundamental matrix of $x' = A(t)x$ with $\Phi(t_0) = I$. **5**

(b) Prove that $x' = A(t)x$ is restrictively stable if and only if it is reducible to zero. **5**

(c) Show that the zero solution of $u' = u(u^2 - 1)$ is asymptotically stable. **4**

OR

3 (a) Let $f(t)$ be a continuous function and $v(t)$ a non-negative continuous function on the interval $t_0 \le t \le t_0 + a$. And if a continuous function $u(t)$ satisfy **5**

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds, \quad t \in [t_0, t_0 + a]$$ then show that $u(t)$

also satisfy the inequality

$$u(t) \le f(t) + \int_{t_0}^{t} f(s)v(s)\exp\left[\int_{s}^{t} v(r)dr\right]ds$$

(b) A system $x't = A(t)x(t)$ is uniformly stable if it is stable and reducible. **5**

(c) For the equation $u' = u(1-u)$ show that the solution $u(t) \equiv 1$ is asymptotically stable. **4**

4 (a) Suppose that the function $F(t, x)$ is continuous in the set $B = \{(t.x)/t_0 \le t \le t_0 + a, \|x - x_0\| \le b\}$ and satisfies the Lipschitz condition $\|F(t, x_1) - F(t\ x_2)\| \le k\|x_1 - x_2\|$ for (t, x_1), $(t, x_2) \in B$ then prove that **5**

$x_1 \to x_2 \Rightarrow x(t, t_0, x_n) \to x(t, t_0, x_0)$ uniformly for $t \in [t_0, t_0 + a]$ where $x(t, t_0, x_n)$ and $x(t, t_0, x_0)$ are solutions of $x'(t) = F(t, x)$ through initial points (t_0, x_n), (t_0, x_0) respectively.

(b) Show that if $x'(t) = A(t)x(t)$ is stable and **5**

$\int_0^t T_r A(s)ds \geq -d > -\infty$ for $t_0 \geq 0$ then it is restrictively stable.

(c) If $f(t)$ is a continuous function on $0 \leq t < \infty$ **4**

and $\int_0^\infty |f(s)| ds$ is finite, show that the solution

$u(t)$ of $u' + (\alpha + f(t))u = 0$ approaches zero at

$t \to \infty; (\alpha > 0)$.

OR

4 (a) Let $\Phi(t)$ be a fundamental metrix of $x'(t) = A(t)x(t)$ **5**

with $\Phi(t_0) = I$. Then prove that $x'(t) = A(t)x(t)$ is uniformly stable if and only if there exists a positive constant M such that $\left\|\Phi(t)\Phi^{-1}(s)\right\| \leq M$, for

$t_0 \leq t < t < \infty$.

(b) Let $g(t)$ and $f(t)$ be any two continuous function **5**

on the interval $a \leq t \leq b$ then the inequality

$$\int_a^b g(s) f(s) ds \leq \left[\int_a^b g^2(s) ds\right]^{1/2} \left[\int_a^b f^2(s) ds\right]^{1/2}$$ holds.

(c) Use Hurwitz's criterian to determinte wether the solutions of the system $x^1 = Ax$ approach zero as $t \to \infty$ or not ? Where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -11 & -5 \end{bmatrix}$ **4**

5 (a) Let $u_1(t)$ and $u_2(t)$ be two linearly independent solutions of $v''+a(t)v=0$ on the interval $0 \le t < \infty$ then prove that the general solution $u(t)$ of the inhomogeneous equation $u''+a(t)v=g(t)$ is given by **5**

$$u(t)=c_1u_1(t)+c_2u_2(t)+\int_0^t\left[u_1(s)u_2(t)-u_1(t)u_2(s)\right]g(s)ds.$$

(b) Show that all the solutions of $u''+e^t u=0$ are bounded. **5**

(c) Use the transformation $u = v^{\exp\left[-\frac{1}{2}\int_0^t p(s)ds\right]}$ to reduce the equation $u''+p(t)u'+q(t)u=0$ to the form $v''+a(t)v=0$. **4**

OR

5 (a) Let $a(t)$ be a continuously differentiable function for $t \in [0,\infty)$. Then prove that all the solutions of $u''+a(t)u=0$ are bounded on $[0,\infty)$ provided that $a(t) \to \infty$ monotonically as $(t) \to \infty$. **5**

(b) If $u = \Phi(t)$ is a solution of the Riccati equation **5**

$u' = p(t)u + q(t)u^2 + r(t)$, show that this equation has

other solution of the form $u = \Phi(t) - \frac{1}{\psi(t)}$ where

$\psi(t)$ is a solution of the equation

$u' = a(t)u + b(t)$.

(c) Determine the type of stability of the critical point **4**

$(0,0)$ of $x_1' = \alpha x_1 + 4x_2,\ x_2' = 3x_1 + x_2$. Also sketch the phase portraits for the same.
